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A biparametric family of optimally convergent sixteenth-order multipoint methods with their fourth-step weighting function as a sum of a rational and a generic two-variable function

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ABSTRACT

A biparametric family of four-step multipoint iterative methods of order sixteen to numerically solve nonlinear equations are developed and their convergence properties are investigated. The efficiency indices of these methods are all found to be $16^{1/5} \approx 1.741101$, being optimally consistent with the conjecture of Kung–Traub. Numerical examples as well as comparison with existing methods developed by Kung–Traub and Neta are demonstrated to confirm the developed theory in this paper.

1. Introduction

To approximate a simple root α of a nonlinear equation $f(x) = 0$, a variety of eighth-order 3-step multipoint iterative methods free from second derivatives have been developed in [1–5]. The efficiency index [6] of these methods is found to be $8^{1/4}$. The 2nd-step of these methods frequently uses King's fourth-order method [7] or Jarratt's fourth-order method [8].

Petković [9] recently has claimed a new development of a general class of optimal r-point methods with convergence order of 2^r. His efficiency index, unfortunately, has turned out to be far from being optimal due to the some unexpected logical errors appearing in (3.9) of page 4406 in [9], which does not yield $(r + 1)$ function evaluations for any $r \geq 4$. Hence the iterative method developed by Petković will not be of further interest to us, being excluded from our discussion here.

We begin our analysis by introducing classical results of Kung and Traub [10] who carried out elegant analyses for two general classes of m-point iterative methods with optimal convergence order of 2^{m-1}. The first class consists of an m-point iteration of f with no evaluation of f' as shown below by (1.1):

$$\begin{cases} \psi_0(f)(x) = x, \\ \psi_1(f)(x) = x + \theta f(x), \quad \text{with } \theta \in \mathbb{R} \text{ as a nonzero constant}, \\ \vdots \\ \psi_{j+1}(f)(x) = Q_j(0), \quad \text{for } j = 1, 2, \ldots, m-1, \end{cases} \tag{1.1}$$

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where $Q_j(y)$ is the inverse interpolatory polynomial for f at $f(\psi_k(f)(x))$, $k = 0, 1, \ldots, j$. That is, $Q_j(y)$ is the polynomial of degree at most j satisfying $Q_j(f(\psi_k(f)(x))) = \psi_k(f)(x)$, $k = 0, 1, \ldots, j$. The corresponding error equation for (1.1) is given by:

with $\quad e_n = x_n - \alpha \quad$ for $n = 0, 1, 2, \ldots$

$$e_{n+1} = S(\psi_m, f)\, e_n^{2^{m-1}} + O(e_n^{1+2^{m-1}}), \tag{1.1a}$$

where $S(\psi_m, f) = Y_m(f)\ \Pi_{k=1}^{m-1} S(\psi_k, f)$ with $Y_m(f) = (-1)^{m+1} \frac{F^{(m)}(0)}{m!F'(0)^m}$, $F = f^{-1}$ (*inverse function of* f) and $S(\psi_1, f) = 1 + \theta f'(\alpha)$. The second class consists of an $(m-1)$-point iteration with $(m-1)$ evaluations of f and one evaluation of f' as shown below by (1.2):

$$\begin{cases} \omega_1(f)(x) = x, \\ \omega_2(f)(x) = x - f(x)/f'(x), \\ \vdots \\ \omega_{j+1}(f)(x) = R_j(0), \quad \text{for } j = 2, 3, \ldots, m-1, \end{cases} \tag{1.2}$$

where $R_j(y)$ is the inverse Hermite interpolatory polynomial [10] of degree at most j satisfying $R_j(f(x)) = x$, $R_j'(f(x)) = 1/f'(x)$, $R_j(f(\omega_k(f)(x))) = \omega_k(f)(x)$, $k = 2, 3, \ldots, j$. The corresponding error equation for (1.2) is given by

$$e_{n+1} = S(\omega_m, f)\, e_n^{2^{m-1}} + O(e_n^{1+2^{m-1}}), \tag{1.2a}$$

where $S(\omega_m, f) = S(\psi_m, f)/[1 + \theta f'(\alpha)]^{2^{m-2}}$.

For the purpose of comparison, we choose $m = 5$ for (1.1), which explicitly yields usual four-step methods as follows:

$$\begin{cases} y_n = x_n + \theta f(x_n), \quad \text{with } \theta \in \mathbb{R} \text{ as a nonzero constant}, \\ z_n = y_n + \mathcal{G}_f(x_n), \\ s_n = z_n + \mathcal{H}_f(x_n) + \mathcal{K}_f(x_n), \\ x_{n+1} = s_n + \mathcal{W}_f(x_n), \end{cases} \tag{1.3}$$

where

$$\begin{aligned} &\mathcal{G}_f(x_n) = \frac{\theta f(x_n) f(y_n)}{f(x_n) - f(y_n)}, \\ &\mathcal{K}_f(x_n) = \mathcal{G}_f(x_n) \frac{f(x_n) f(z_n)}{[f(x_n) - f(z_n)][f(y_n) - f(z_n)]}, \\ &T_f(x_n) = \mathcal{K}_f(x_n) \frac{f(q_n)}{[f(x_n) - f(q_n)][f(y_n) - f(q_n)][f(z_n) - f(q_n)]}, \quad q_n = z_n + \mathcal{K}_f(x_n), \\ &\mathcal{H}_f(x_n) = T_f(x_n)[f(x_n) f(y_n) + f(z_n)^2 - f(z_n) f(q_n)], \\ &h_1 = (f(z_n)(f(z_n) - f(s_n)) + f(x_n) f(y_n)) f(z_n), \\ &h_2 = f(q_n)(f(q_n) - f(s_n))(-f(x_n) - f(y_n) + f(q_n) + f(s_n)), \\ &t_1 = f(x_n) f(y_n)(h_1 - h_2) + f(z_n) f(q_n)(f(z_n) - f(q_n))(f(z_n) - f(s_n))(f(q_n) - f(s_n)), \\ &t_2 = (f(x_n) - f(s_n))(f(y_n) - f(s_n))(f(z_n) - f(s_n))(f(q_n) - f(s_n)), \\ &\mathcal{W}_f(x_n) = T_f(x_n)\, f(s_n)\, t_1/t_2. \end{aligned} \tag{1.3a}$$

The corresponding error equation for (1.3) is given by

$$e_{n+1} = c_2^4(2c_2^2 - c_3)^2(5c_2^3 - 5c_2c_3 + c_4)(14c_2^4 - 21c_2^2c_3 + 3c_3^2 + 6c_2c_4 - c_5)(1 + \theta f'(\alpha))^8\, e_n^{16} + O(e_n^{17}). \tag{1.3b}$$

Similarly, we choose $m = 5$ for (1.2), which explicitly yields the usual four-step methods as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \mathbb{G}_f(x_n), \\ s_n = z_n - \mathbb{K}_f(x_n), \\ x_{n+1} = s_n - \mathbb{W}_f(x_n), \end{cases} \tag{1.4}$$

where

$$\begin{aligned} &\mathbb{G}_f(x_n) = \frac{f(x_n)^2 f(y_n)}{f'(x_n)[f(x_n) - f(y_n)]^2}, \\ &\mathbb{K}_f(x_n) = \mathbb{G}_f(x_n) \frac{[f(x_n)^2 + f(y_n)^2 - f(y_n) f(z_n)]}{[f(x_n) - f(z_n)]^2[f(y_n) - f(z_n)]}, \\ &h_0 = f(y_n)[f(x_n)^2 - f(s_n) f(y_n) + f(y_n)^2] + f(z_n)[f(s_n) - f(z_n))(f(s_n) - 2f(x_n) + f(z_n)], \\ &h_1 = \mathbb{G}_f(x_n) f(s_n) f(z_n)[h_0 f(x_n)^2 + f(y_n) f(z_n)[f(y_n) - f(s_n)][f(y_n) - f(z_n)][f(z_n) - f(s_n)]], \\ &h_2 = [f(x_n) - f(s_n)]^2[f(y_n) - f(s_n)][f(x_n) - f(z_n)]^2[f(y_n) - f(z_n)][f(z_n) - f(s_n)], \\ &\mathbb{W}_f(x_n) = h_1/h_2. \end{aligned} \tag{1.4a}$$

The corresponding error equation for (1.4) is given by

$$e_{n+1} = c_2^4(2c_2^2 - c_3)^2(5c_2^3 - 5c_2c_3 + c_4)(14c_2^4 - 21c_2^2c_3 + 3c_3^2 + 6c_2c_4 - c_5)\, e_n^{16} + O(e_n^{17}). \tag{1.4b}$$

In 1981, Neta [11] also suggested a family of sixteenth-order multipoint iterative methods which are introduced here:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + Af(y_n)}{f(x_n) + (A-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \quad A \in \mathbb{R}, \\ s_n = y_n + \delta_1 f^2(x_n) + \delta_2 f^3(x_n), \\ x_{n+1} = y_n + \theta_1 f^2(x_n) + \theta_2 f^3(x_n) + \theta_3 f^4(x_n), \end{cases} \tag{1.5}$$

where $\delta_2 = -\frac{\phi_y - \phi_z}{F_y - F_z}$, $\delta_1 = \phi_y + \delta_2 F_y, \theta_3 = \frac{\Delta_1 - \Delta_2}{F_s - F_y}, \theta_2 = -\Delta_1 + \theta_3(F_s + F_z), \theta_1 = \phi_s + \theta_2 F_s - \theta_3 F_s^2$ with $\Delta_1 = \frac{\phi_s - \phi_z}{F_s - F_z}$, $\Delta_2 = \frac{\phi_y - \phi_z}{F_y - F_z}$, $\phi_s = \frac{1}{F_s}(\frac{s_n - x_n}{F_s} - \frac{1}{f'(x_n)})$, $\phi_y = \frac{1}{F_y}(\frac{y_n - x_n}{F_y} - \frac{1}{f'(x_n)})$, $\phi_z = \frac{1}{F_z}(\frac{z_n - x_n}{F_z} - \frac{1}{f'(x_n)})$, $F_s = f(s_n) - f(x_n)$, $F_y = f(y_n) - f(x_n)$ and $F_z = f(z_n) - f(x_n)$.

Notice that the fourth equation of (1.5) is obtained by means of inverse interpolation [6] and the coefficients δ_i $(i = 1, 2)$ as well as θ_i $(i = 1, 2, 3)$ are dependent upon the values of $x_n, y_n, z_n, s_n f(x_n), f(y_n), f(z_n), f(s_n), f'(x_n)$. Such a function-dependent scheme unfavorably requires much computational time. Although Neta did not provide an explicit form of the error equation of (1.5), we successfully find the corresponding error equation as follows:

$$e_{n+1} = c_2^4[(1 + 2A)c_2^2 - c_3]^2(5c_2^3 - 5c_2c_3 + c_4)(14c_2^4 - 21c_2^2c_3 + 3c_3^2 + 6c_2c_4 - c_5)\, e_n^{16} + O(e_n^{17}). \tag{1.6}$$

Due to the extensive use of forward divided differences, iterative methods (1.3)–(1.5) all require a large amount of overall function evaluations, even though they have only five new function evaluations per iteration. As a result, they have the disadvantage that the corresponding computational time will be increased as compared with methods using less forward divided differences.

The main aim is to develop a fast sixteenth-order method which is free of forward divided differences and optimally convergent in accordance with the conjecture of Kung–Traub [10] for complex-valued as well as real-valued nonlinear equations. We assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic [12] in a region containing α. To avoid the use of forward divided differences, we introduce constant control parameters to propose a new family of four-step multipoint methods described as follows: for $n = 0, 1, \ldots,$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(u_n)\dfrac{f(y_n)}{f'(x_n)}, \\ s_n = z_n - H_f(u_n, v_n, w_n)\dfrac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = s_n - W_f(u_n, v_n, w_n, t_n)\dfrac{f(s_n)}{f'(x_n)}, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} K_f(u_n) &= \frac{1 + \beta u_n + \left(-9 + \frac{5\beta}{2}\right) u_n^2}{1 + (\beta - 2)u_n + (-4 + \beta 2)\, u_n^2}, \\ H_f(u_n, v_n, w_n) &= \frac{1 + 2u_n + (2 + \sigma)w_n}{1 - v_n + \sigma w_n}, \\ W_f(u_n, v_n, w_n, t_n) &= \frac{1 + 2u_n + (2 + \sigma)v_n w_n}{1 - v_n - 2w_n - t_n + 2(1 + \sigma)v_n w_n} + G(u_n, w_n) \end{aligned} \tag{1.8}$$

are respectively called the second-step, the third-step and the fourth-step weighting functions; $G : \mathbb{C}^2 \to \mathbb{C}$ is an analytic function in a region containing the origin $(0, 0)$; two constant real parameters β, σ are to be chosen freely and the following notations are used

$$u_n = f(y_n)/f(x_n), \qquad v_n = f(z_n)/f(y_n), \qquad w_n = f(z_n)/f(x_n), \qquad t_n = f(s_n)/f(z_n). \tag{1.9}$$

It is not difficult to show that

$$y_n = \alpha + O(e_n^2), \qquad z_n = \alpha + O(e_n^4), \qquad s_n = \alpha + O(e_n^8), \tag{1.10}$$

$$u_n = O(e_n), \qquad v_n = O(e_n^2), \qquad w_n = O(e_n^3), \qquad t_n = O(e_n^4), \tag{1.11}$$

$$\frac{f(x_n)}{f'(x_n)} = O(e_n), \qquad \frac{f(y_n)}{f'(x_n)} = O(e_n^2), \qquad \frac{f(z_n)}{f'(x_n)} = O(e_n^4), \qquad \frac{f(s_n)}{f'(x_n)} = O(e_n^8). \tag{1.12}$$

We now introduce general constant parameters $\lambda, \mu, \gamma, a, b, c, d$ in two expressions of (1.8) to get:

$$K_f(u) = \frac{1+\beta u+\lambda u^2}{1+(\beta-2)u+\mu u^2}, \quad H_f(u,v,w) = \frac{1+au+bv+\gamma w}{1+cu+dv+\sigma w}. \tag{1.13}$$

Similarly, we introduce general constant parameters $B_1, B_2, \ldots, B_6$ for W_f in (1.8) to take the form as a sum of a rational and a generic two-variable function to be seen below:

$$W_f(u,v,w,t) = \frac{1+B_1u+B_2vw}{1+B_3v+B_4w+B_5t+B_6vw} + G(u,w). \tag{1.14}$$

Then through an analysis to be shown in Section 2, the desired form of W_f in (1.8) will be obtained along with the derivation of the corresponding error equation stating convergence order of sixteen.

Observe that (1.7) requires five new function evaluations for $f(x_n), f(y_n), f(z_n), f(s_n)$ and $f'(x_n)$ per iteration. Further analysis with this observation will lead to the development of a new family of sixteenth-order method having its efficiency index [5] as $16^{1/5} \approx 1.741101$. To measure convergence behavior within a given error bound, the values of $|x_n - \alpha|$ as well as CPU times of proposed methods (1.7) with (1.3)–(1.5) will be compared. Typical forms of $W_f(u_n, v_n, w_n, t_n)$ are displayed in Section 2. Numerical examples are presented in Section 3 to verify the underlying theory developed in this paper.

2. Convergence analysis

In what follows, Theorem 2.1 describes the convergence analysis on iterative scheme (1.7) with (1.8).

Theorem 2.1. *Suppose that $G : \mathbb{C}^2 \to \mathbb{C}$ is an analytic function in a region containing the origin $(0,0)$ and that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic in a region containing α. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$. Let $G_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}G(u,w)}{\partial u^i \partial w^j}\Big|_{(0,0)}$ for $i, j = 0, 1, \ldots$. Let $c_2c_3c_4 \neq 0$ and x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let β and σ be real constant parameters to be chosen freely. If the following relations in (1.13) and (1.14) hold*

$$\begin{cases} a=2, \quad b=0, \quad c=0, \quad d=-1, \quad \gamma=2+\sigma, \quad \lambda=-9+5\beta/2, \quad \mu=-4+\beta/2, \\ B_1=2, \quad B_2=2+\sigma, \quad B_3=-1, \quad B_4=-2, \quad B_5=-1, \quad B_6=2(1+\sigma), \\ G_{00}=G_{01}=G_{10}=G_{20}=G_{30}=G_{40}=G_{50}=G_{60}=G_{70}=0, \quad G_{21}=-6, \quad G_{02}=-6-4\sigma, \\ G_{11}=-3-2\sigma, \quad G_{12}=2(\sigma^2-2\sigma-9), \quad G_{31}=-12+\dfrac{11}{2}\beta, \quad G_{41}=-68+33\beta-\dfrac{11}{2}\beta^2, \end{cases} \tag{2.1}$$

then iterative scheme (1.7) defines a biparametric family of sixteenth-order methods satisfying the error equation below: with $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$

$$e_{n+1} = \frac{1}{8}c_2^3(6c_2^2-c_3)\{-2c_4+c_2^3(-76+11\beta-24\sigma)+4c_2c_3(4+\sigma)\}\phi\, e_n^{16} + O(e_n^{17}), \tag{2.2}$$

where $\phi = 4c_3^2c_4 + 48c_2^4c_4(2+\sigma) - 8c_2^2c_3c_4(5+\sigma) + c_2^5c_3\{-48G_{22} - 4G_{51} + 11\beta(118+\beta(-17+2\beta)) + 44\beta\sigma + 240\sigma^2 - 4(2477+916\sigma)\} + 4c_2^3\{6c_5 + c_3^2(327+G_{22}-11\beta+2(66-7\sigma)\sigma)\} + 4c_2\{-c_3c_5 + c_3^3(-15+(-6+\sigma)\sigma)\} + 2c_2^7\{72G_{22} + 12G_{51} + 2G_{80} - 3(11\beta(94+\beta(-17+2\beta)) + 44\beta\sigma + 48\sigma^2 - 20(211+68\sigma))\}$.

Proof. Taylor series expansion of $f(x_n)$ about α up to sixteenth-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = f'(\alpha)\left\{e_n + \sum_{i=2}^{16} c_i e_n^i + O(e_n^{17})\right\}. \tag{2.3}$$

For ease of notation, e_n will be denoted by e for the time being. With the aid of symbolic computation of Mathematica, a lengthy algebraic computation induces relations (2.4)–(2.9) below:

$$f'(x_n) = f'(\alpha)\left\{1 + \sum_{i=2}^{16} i\, c_i e^{i-1} + O(e^{16})\right\}. \tag{2.4}$$

$$\frac{f(x_n)}{f'(x_n)} = e - c_2e^2 + 2(c_2^2-c_3)e^3 - (4c_2^3-7c_2c_3+3c_4)e^4 + (8c_2^4-20c_2^2c_3+6c_3^2+10c_2c_4-4c_5)e^5 + H_6e^6 + H_7e^7 + H_8e^8 + \sum_{i=9}^{16} H_ie^i + O(e^{17}), \tag{2.5}$$

where $H_i = H_i(c_2, c_3, \ldots, c_i)$ are given in terms of $c_2, c_3, \ldots, c_i$ with explicitly written three coefficients $H_6 = -16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6$, $H_7 = 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)$, $H_8 = -64c_2^7 + 304c_2^5c_3 - 176c_2^4c_4 - 75c_3^2c_4 + 31c_4c_5 + c_2^3(-408c_3^2 + 92c_5) + 4c_2^2(87c_3c_4 - 11c_6) + 27c_3c_6 + c_2(135c_3^3 - 64c_4^2 - 118c_3c_5 + 19c_7) - 7c_8$.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e^4 \\ &\quad - (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e^5 - \sum_{i=6}^{16} H_ie^i + O(e^{17}). \end{aligned} \tag{2.6}$$

$$\begin{aligned} f(y_n) &= f'(\alpha)\left\{c_2e^2 + (2c_3 - 2c_2^2)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 \right. \\ &\quad \left. -\ 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e^5 + \sum_{i=6}^{16} \theta_ie^i + O(e^{17})\right\}, \end{aligned} \tag{2.7}$$

where $\theta_i = \theta_i(c_2, c_3, c_4, c_5, H_6, H_7, \ldots, H_i)$ are given in terms of $c_2, c_3, c_4, c_5, H_6, H_7, \ldots, H_i$ with explicitly written three coefficients $\theta_6 = 12c_2^5 - 21c_2^3c_3 + 4c_2c_3^2 + 6c_2^2c_4 - H_6$, $\theta_7 = -32c_2^6 + 78c_2^4c_3 - 34c_2^2c_3^2 - 32c_2^3c_4 + 12c_2c_3c_4 + 8c_2^2c_5 - H_7$ and $\theta_8 = 48c_2^7 - 144c_2^5c_3 + 65c_2^4c_4 + 4c_2^3(27c_3^2 - 4c_5) + c_2(-12c_3^3 + 9c_4^2 + 16c_3c_5) - c_2^2(73c_3c_4 + 2H_6) - H_8$. Using K_f in (1.13), we obtain:

$$z_n = y_n - K_f(x_n)\frac{f(y_n)}{f'(x_n)} = \alpha + \sum_{i=4}^{16} L_ie^i + O(e^{17}), \tag{2.8}$$

where coefficients $L_i = L_i(c_2, c_3, \ldots, c_6, \lambda, \beta, \mu, H_6, H_7, \ldots, H_{16})$ are given in terms of $c_2, c_3, \ldots, c_6, \lambda, \beta, \mu, H_6, H_7, \ldots, H_{16}$; only four of them are explicitly written by $L_4 = -c_2c_3 + c_2^3(1 + 2\beta - \lambda + \mu)$, $L_5 = -2c_3^2 - 2c_2c_4 + 2c_2^2c_3(4 + 6\beta - 3\lambda + 3\mu) - c_2^4(4 + 2\beta^2 - 8\lambda + 6\mu + \beta(12 - \lambda + \mu))$, $L_6 = -7c_3c_4 + 3c_2^2c_4(4 + 6\beta - 3\lambda + 3\mu) - 2c_2^3c_3\rho_1 + 3c_2\rho_2 + c_2^5\rho_3$, with $\rho_1 = 15 + 8\beta^2 - 29\lambda + 21\mu + \beta(42 - 4\lambda + 4\mu)$, $\rho_2 = (-c_5 + c_3^2(6 + 8\beta - 4\lambda + 4\mu))$, $\rho_3 = 10 + 2\beta^3 + \lambda(-40 + \mu) + 22\mu - \mu^2 + \beta^2(14 - \lambda + \mu) + \beta(44 - 9\lambda + 5\mu)$, $L_7 = 2c_2(3c_6 + H_6 + 9c_3c_4(1 + 4\beta - 2\lambda + 2\mu)) - 4c_2^3c_4\tau_1 + 2\tau_2 + 2c_2^4c_3\tau_3 - 2c_2^2\tau_4 - c_2^6\tau_5$, with $\tau_1 = -4 + 6\beta^2 - 21\lambda + 15\mu + \beta(30 - 3\lambda + 3\mu)$, $\tau_2 = -3c_4^2 - 5c_3c_5 + c_3^3(6 + 8\beta - 4\lambda + 4\mu)$, $\tau_3 = -12 + 10\beta^3 + 5\lambda(-34 + \mu) + 88\mu - 5\mu^2 + \beta^2(62 - 5\lambda + 5\mu) + \beta(176 - 41\lambda + 21\mu)$, $\tau_4 = c_5(5 - 12\beta + 6\lambda - 6\mu) + c_3^2(7 + 24\beta^2 - 78\lambda - 12\beta(-9 + \lambda - \mu) + 54\mu)$, $\tau_5 = 2\beta^4 + \beta^3(16 - \lambda + \mu) + \beta^2(56 - 10\lambda + 4\mu) + 2\beta(64 + \lambda(-24 + \mu) + 4\mu - \mu^2) + 2(-6 + 32\mu - 5\mu^2 + \lambda(-80 + 6\mu))$, and the remaining L_i's are omitted for economical paper space.

Notice that L_5 and L_6 are free of H_i $(i = 6, 7, \ldots, 16)$ while L_7 contains a single H_6-term but L_8 contains both H_6-term and H_7-term. Similarly L_k $(k = 7, 8, \ldots, 16)$ contains $k - 6$ terms with $H_6, H_7, \ldots, H_{k-2}, H_{k-1}$ as its respective factors, from the construction of L_k defined recursively on H_j's. We also find

$$f(z_n) = f'(\alpha)\left\{L_4e^4 + L_5e^5 + L_6e^6 + L_7e^7 + (c_2L_4^2 + L_8)e^8 + \sum_{i=9}^{16} Z_ie^i + O(e^{17})\right\}, \tag{2.9}$$

where $Z_i = Z_i(L_4, L_5, \ldots, L_{16})$ are given in terms of $L_4, L_5, \ldots, L_{16}$. Using relations (2.3)–(2.9), we can further express u_n, v_n and w_n in terms of $\beta, \lambda, \mu, a, b, c, d, \gamma, \sigma$ and c_j $(j = 2, 3, \ldots, 16)$ by the aid of symbolic computation of Mathematica to compute s_n in (1.7) with H_f in (1.13):

$$s_n = \alpha + S_4e^4 + S_5e^5 + S_6e^6 + S_7e^7 + S_8e^8 + \sum_{i=9}^{16} S_ie^i + O(e^{17}), \tag{2.10}$$

where S_i $(i = 4, 5, \ldots, 16)$ are multivariate polynomials in H_k $(6 \le k \le 16)$, L_ν $(4 \le \nu \le 16)$, $\beta, \lambda, \mu, a, b, c, d, \gamma, \sigma$ and c_j $(j = 2, 3, 4, 5)$ or $\lambda, \beta, \mu, a, b, c, d, \gamma, \sigma$ and c_j $(j = 2, 3, \ldots, 16)$; for instance, $S_4 = -c_2c_3 - L_4 + c_2^3(1 + 2\beta - \lambda + \mu) = 0$ is satisfied with $L_4 = -c_2c_3 + c_2^3(1 + 2\beta - \lambda + \mu)$ in (2.7) and

$$S_5 = (2 - a + c)\, c_2^2\{-c_3 + c_2^2(1 + 2\beta - \lambda + \mu)\}. \tag{2.11}$$

We seek relations among constant control parameters by requiring $S_5 = S_6 = S_7 = \cdots = 0$ to achieve maximal order of convergence. The requirement $S_5 = 0$ immediately yields

$$c = a - 2. \tag{2.12}$$

As a result, coefficients $S_6 = S_7 = 0$ with $c = a - 2$ regardless of c_j's yield relations below:

$$d = b - 1, \qquad \lambda = 2\beta + \mu - 2a - 1, \qquad \gamma = a + 2b + \sigma, \qquad \mu = -a^2 + \left(a - \frac{3}{2}\right)\beta. \tag{2.13}$$

Additionally coefficient S_8 in conjunction with (2.13) yields the following relation:

$$S_8 = \frac{1}{2}c_2\{2(1+a)c_2^2 - c_3\}\omega_0, \tag{2.14}$$

with $\omega_0 = 2bc_3^2 + 2c_2c_4 - 4c_2^2c_3\{4b + a(2+b) + \sigma\} + c_2^4\{-4 + 24b - 3\beta + 8\sigma + 2a(6 + 7a + 12b - 2\beta + 4\sigma)\}$. Although the original form of S_8 contains a term $2(2-a+c)c_2^2H_6$, the imposed relation $c = a - 2$ makes it being independent of coefficient H_6. Coefficient S_8 can no longer be set to zero independently of c_j's, since it contains terms c_j's with their factors free from control parameters. Note also that S_9 is free of H_i $(i = 6, 7, \ldots, 16)$, while S_{10} contains a single H_6-term but S_{11} contains both an H_6-term and an H_7-term. Similarly S_k $(k = 12, \ldots, 16)$ contains $k - 9$ terms with $H_6, H_7, \ldots, H_{k-3}, H_{k-4}$ as its respective factors, in view of the recursive construction of S_k with respect to H_j's.

Using relations (2.3)–(2.14), we can further express u_n, v_n, w_n and t_n in terms of a, b, β, σ and c_j $(j = 2, 3, \ldots, 16)$ by the aid of symbolic computation of Mathematica to compute x_{n+1} in (1.7) with W_f in (1.14) as follows:

$$x_{n+1} = \alpha + h_8e^8 + h_9e^9 + h_{10}e^{10} + h_{11}e^{11} + h_{12}e^{12} + h_{13}e^{13} + h_{14}e^{14} + h_{15}e^{15} + h_{16}e^{16} + O(e^{17}), \tag{2.15}$$

where h_i $(i = 8, 9, \ldots, 16)$ are multivariate polynomials in $\lambda, \beta, \mu, a, b, \gamma, \sigma, B_j$ $(j = 1, 2, \ldots, 6)$, S_j $(j = 8, 9, \ldots, 16)$ and c_j $(j = 2, 3, \ldots, 16)$; for instance,

$$h_8 = -G_{00}\,S_8, \qquad h_9 = (2 - B_1 + G_{10})c_2S_8. \tag{2.16}$$

We impose conditions $h_8 = h_9 = h_{10} = \cdots = h_{14} = h_{15} = 0$ and $h_{16} \neq 0$ independently of c_j's so that the iterative scheme (1.7) has sixteenth-order convergence. Requiring $h_8 = h_9 = h_{10} = h_{11} = 0$ independently of c_j's gives us 6 independent relations, from which we find $G_{00}, G_{10}, B_3, G_{20}, B_4, G_{30}$ as follows:

$$\begin{aligned} &G_{00} = 0, \qquad G_{10} = 2 - B_1, \qquad B_3 = -1, \qquad G_{20} = -2(a-2), \\ &B_4 = -4 + B_1 + G_{01}, \qquad G_{30} = 2(a-2)^2. \end{aligned} \tag{2.17}$$

Substituting (2.17) into $h_{12} = 0$ with λ, μ in (2.13), L_5, L_6 in (2.7) after simplification yields

$$h_{12} = -\frac{1}{2}\,S_8\,A = 0, \tag{2.18}$$

with $A = -2bB_5c_3^2 - 2(1+B_5)c_2c_4 + 2c_2^2c_3(13 + 8bB_5 + 2a(-1 + (2+b)B_5) + B_1(-4 + B_1 + G_{01}) - G_{11} + 2B_5\sigma) + c_2^4(-80 + 4a^3 - 14a^2(1+B_5) - 4B_1(-4 + B_1 + G_{01}) + 4G_{11} + 2G_{40} + 3\beta + B_5(4 - 24b + 3\beta - 8\sigma) - 4a(-2 + 3B_5 + 6bB_5 + B_1(-4 + B_1 + G_{01}) - G_{11} - \beta - B_5\beta + 2B_5\sigma))$.

Hence setting $A = 0$ to make $h_{12} = 0$, independently of c_2, c_3 and c_4, yields the following relations:

$$B_5 = -1, \qquad b = 0, \qquad G_{11} = 13 - 6a + (G_{01} - 4)B_1 + B_1^2 - 2\sigma, \qquad G_{40} = -2(a-2)^3. \tag{2.19}$$

Substituting these $B_3, B_4, B_5, b, G_{00}, G_{10}, G_{11}, G_{20}, G_{30}, G_{40}$ into $h_{13} = 0$ with λ, μ in (2.13), L_5, L_6, L_7 in (2.7) and S_8, S_9 in (2.14) after simplification yields

$$h_{13} = \frac{1}{4}c_2^3\{2(1+a)c_2^2 - c_3\}\omega_1\,B = 0, \tag{2.20}$$

with $\omega_1 = 2c_4 - 4c_2c_3(2a + \sigma) + c_2^3(-4 + 14a^2 - 3\beta - 4a(-3 + \beta - 2\sigma) + 8\sigma)$ and $B = -2(-2 + B_1)c_2c_4 - 2c_3^2(a - B_1 + B_2 - B_6 - 2G_{01} + \sigma) - 2c_2^2c_3(34 + 4B_1 - 4B_2 + 4B_6 + 8G_{01} + 4a(-8 + a - B_2 + B_6 + 2G_{01}) - G_{21} - 2(4 + B_1)\sigma) + c_2^4(192 - 8a^3 + 4a^4 + 12B_1 - 8B_2 + 8B_6 + 16G_{01} - 4G_{21} - 2G_{50} - 6\beta + 3B_1\beta - 8(3 + B_1)\sigma + a^2(-4 - 6B_1 - 8B_2 + 8B_6 + 16G_{01} + 8\sigma) + 4a(-30 + B_1 - 4B_2 + 4B_6 + 8G_{01} - G_{21} - 2\beta + B_1\beta - 2(2 + B_1)\sigma))$.

In order to make $h_{13} = 0$ independently of c_2, c_3 and c_4, we set $B = 0$ in (2.20) to obtain:

$$B_1 = 2, \qquad G_{01} = \frac{1}{2}(B_2 - B_6 + a - 2\sigma), \qquad G_{21} = 34 + 8a^2 + 4a(\sigma - 9) - 8\sigma, \qquad G_{50} = 2(a-2)^4. \tag{2.21}$$

Substituting these $B_1, B_3, B_4, B_5, b, G_{00}, G_{01}, G_{10}, G_{11}, G_{20}, G_{21}, G_{30}, G_{40}, G_{50}$ into $h_{14} = 0$ with λ, μ in (2.13), L_5, L_6, L_7, L_8 in (2.7) and S_8, S_9, S_{10} associated with (2.14) after simplification yields

$$h_{14} = -\frac{1}{8}c_2^4\{2(1+a)c_2^2 - c_3\}\,\omega_1\,C = 0, \tag{2.22}$$

with ω_1 described as in (2.20) and $C = 8(-2+a)c_2c_4 + c_3^2(-3a^2 + 2a(26 + B_2 - B_6 - \sigma) + (B_2 - B_6 + \sigma)^2 + 4(-13 - 5B_2 + 3B_6 + G_{02} + 5\sigma)) + 4c_2^4(-276 - 3a^4 + 2a^5 - 20B_2 + 12B_6 + 4G_{02} + 2G_{31} + G_{60} + 3\beta + 52\sigma + (B_2 - B_6 + \sigma)^2 + 2a^3(15 + B_2 - B_6 + 5\sigma) + a^2(19 + B_2^2 + 8B_6 + B_6^2 + 4G_{02} - 8\beta - 2B_2(8 + B_6 - \sigma) - 2(6 + B_6)\sigma + \sigma^2) + 2a(94 + B_2^2 + 11B_6 + 4G_{02} + G_{31} - \beta + (B_6 - \sigma)^2 + 15\sigma + B_2(-19 - 2B_6 + 2\sigma))) - 2c_2^2c_3(-272 - 40B_2 + 24B_6 + 8G_{02} + 2G_{31} - 3\beta + 72\sigma + 2(7a^3 + (B_2 - B_6 + \sigma)^2 + 2a^2(-4 + B_2 - B_6 + 2\sigma) + a(B_2^2 + B_6(10 + B_6) + 4G_{02} - 2B_2(9 + B_6 - \sigma) - 2B_6\sigma + \sigma^2 - 2(-61 + \beta + \sigma))))$.

Table 1
Various methods with typical choices of β, σ and $G(u, w)$.

Case	Method	(β, σ)	$G(u, w)$
1	**Y1**	$(2, -2)$	$-\frac{1}{2}[uw\{6+12u+u^2(24-11\beta)+u^3\phi_1{}^{a}+4\sigma\}]+\phi_2{}^{b}w^2$
2	**Y2**	$(\frac{18}{5}, -1)$	$-\frac{1}{2}[u\ \sin w\{6+12u+u^3\phi_1+u^2(24-11\beta)+4\sigma\}]+\phi_2\ w^2$
3	**Y3**	$(0, -2)$	$-\frac{1}{2}[uw\{6+12u+u^2(24-11\beta)+u^3\phi_1+4\sigma\}]+\phi_2\ w\ \sin w$
4	**Y4**	$(2, -1)$	$-\frac{1}{2}[uw\{6+12u+u^2(24-11\beta)+u^3\phi_1+4\sigma\}]+\phi_2\ \sin^2 w$
5	**Y5**	$(0, 0)$	$-\frac{1}{2}[u\ \sinh w\{6+12u+u^2(24-11\beta)+u^3\phi_1+4\sigma\}]+\phi_2\ w^2$

[a] $\phi_1 = 11\beta^2 - 66\beta + 136$.
[b] $\phi_2 = 2u(\sigma^2 - 2\sigma - 9) - 4\sigma - 6$.

In order to make $h_{14} = 0$ independently of c_2, c_3 and c_4, we set $C = 0$ in (2.22) to obtain:

$$\begin{aligned} &a = 2, \qquad G_{02} = \frac{1}{4}[-B_2^2 - (B_6-\sigma)^2 + 2B_2(B_6-\sigma+8) - 8(B_6+2\sigma+5)], \\ &G_{31} = \frac{11\beta}{2} - 12, \qquad G_{60} = 0. \end{aligned} \tag{2.23}$$

Substituting these $B_1, B_3, B_4, B_5, a, b, G_{00}, G_{01}, G_{02}, G_{10}, G_{11}, G_{20}, G_{21}, G_{30}, G_{31}, G_{40}, G_{50}, G_{60}$ into $h_{15} = 0$ with λ, μ in (2.13), L_5, L_6, L_7, L_8, L_9 in (2.7) and S_8, S_9, S_{10}, S_{11} associated with (2.14) after simplification yields

$$h_{15} = \frac{1}{4}c_2^3(6c_2^2 - c_3)\{-2c_4 + c_2^3(-76 + 11\beta - 24\sigma) + 4c_2c_3(4+\sigma)\}\, D = 0, \tag{2.24}$$

with $D = c_3^3(B_2 + B_6 - 3\sigma - 4) - 12c_2^3c_4(B_2 - B_6 + \sigma) + 2c_2c_3c_4(B_2 - B_6 + \sigma) + c_2^2c_3^2(108 + B_2^2 + B_6^2 + 2G_{12} + 38\sigma - 7\sigma^2 + 2B_6(\sigma+3) - 2B_2(B_6+\sigma+21)) - c_2^4c_3(1000 + 12B_2^2 + 12B_6^2 + 24G_{12} + 2G_{41} - 66\beta + 11\beta^2 + 152\sigma + 11\beta\sigma - 84\sigma^2 + B_6(160 - 11\beta + 24\sigma) - B_2(24B_6 - 11\beta + 24\sigma + 376)) + 2c_2^6(1488 + 18B_2^2 + 18B_6^2 + 36G_{12} + 6G_{41} + G_{70} - 198\beta + 33\beta^2 + 96\sigma + 33\beta\sigma - 126\sigma^2 - 3B_2(12B_6 - 11\beta + 12\sigma + 160) + B_6(264 - 33\beta + 36\sigma))$.

In order to make $h_{15} = 0$ independently of c_2, c_3 and c_4, we set $D = 0$ in (2.24) to obtain:

$$B_6 = 2(\sigma+1), \qquad B_2 = \sigma + 2, \qquad G_{12} = 2(\sigma^2 - 2\sigma - 9), \qquad G_{70} = 0, \qquad G_{41} = 33\beta - \frac{11}{2}\beta^2 - 68. \tag{2.25}$$

Overall, in view of (2.11)–(2.25), we find the desired relation (2.1) among the constant parameters with two free control parameters β, σ. We restore notation e back to e_n in (2.17) and compute h_{16} using (2.12)–(2.25), $L_5, L_6, L_7, L_8, L_9, L_{10}$ in (2.7) and $S_8, S_9, S_{10}, S_{11}, S_{12}$ associated with (2.14) after simplification as follows:

$$h_{16} = \frac{1}{8}c_2^3(6c_2^2 - c_3)\{-2c_4 + c_2^3(-76 + 11\beta - 24\sigma) + 4c_2c_3(4+\sigma)\}\phi, \tag{2.26}$$

where $\phi = 4c_3^2c_4 + 48c_2^4c_4(2+\sigma) - 8c_2^2c_3c_4(5+\sigma) + c_2^5c_3\{-48G_{22} - 4G_{51} + 11\beta(118 + \beta(-17+2\beta)) + 44\beta\sigma + 240\sigma^2 - 4(2477 + 916\sigma)\} + 4c_2^3\{6c_5 + c_3^2(327 + G_{22} - 11\beta + 2(66 - 7\sigma)\sigma)\} + 4c_2\{-c_3c_5 + c_3^3(-15 + (-6+\sigma)\sigma)\} + 2c_2^7\{72G_{22} + 12G_{51} + 2G_{80} - 3(11\beta(94 + \beta(-17+2\beta)) + 44\beta\sigma + 48\sigma^2 - 20(211 + 68\sigma))\}$. This yields the desired relation (1.7) and (2.2) with two free control parameters β, σ, completing the proof. □

Table 1 below lists various methods derived from (1.7) based on some interesting choices of constant control parameters β, σ and the function $G(u, w)$.

3. Algorithm, numerical results and discussions

In view of the analysis described in Section 2, we develop a zero-finding algorithm to be implemented with *Mathematica* [13] as follows:

Algorithm 3.1 (*Zero-Finding Algorithm*).

Step 1. Construct iteration scheme (1.7) with the given function f having a simple zero α for $n \in \mathbb{N} \cup \{0\}$ as mentioned in Section 1.

Step 2. Set the minimum number of precision digits. With exact or most accurate zero α, supply the theoretical asymptotic error constant η, order of convergence p as well as c_2, c_3, c_4 β, σ stated in Section 2. Set the error bound ϵ, the maximum iteration number n_{max} and the initial guess x_0. Compute $|f(x_0)|$ and $|x_0 - \alpha|$.

Step 3. Tabulate the computed values of $n, x_n, |f(x_n)|, |e_n| = |x_n - \alpha|, |\frac{e_n}{e_{n-1}{}^p}|$ and η.

Numerical experiments have been performed with the minimum number of precision digits chosen as 1000, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. For the sake of accurate computation of asymptotic error constants, the zero α, however, was given with

Table 2
Convergence for $f(x) = \cos\frac{\pi x}{2} + x^2 - \pi$ with $\alpha \approx 2.034724896279126$.

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^{16}}\rvert$	η
	0	1.87	0.623915	0.164725		
Y1	1	2.03472489628232	1.32705×10^{-11}	3.19380×10^{-12}	10.86824814	0.02889866001
	2	2.03472489627913	1.40727×10^{-185}	$\mathbf{3.38685 \times 10^{-186}}$	0.02889866001	
	3	2.03472489627913	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.87	0.623915	0.164725		
Y2	1	2.03472489625302	1.08482×10^{-10}	2.61083×10^{-11}	88.84459859	3.173905206
	2	2.03472489627913	6.14663×10^{-169}	1.47930×10^{-169}	3.173905208	
	3	2.03472489627913	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.87	0.623915	0.164725		
Y3	1	2.03472489604356	9.78809×10^{-10}	2.35569×10^{-10}	801.6223004	14.60170407
	2	2.03472489627913	5.45590×10^{-153}	1.31307×10^{-153}	14.60170415	
	3	2.03472489627913	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.87	0.623915	0.164725		
Y4	1	2.03472489586462	1.72230×10^{-9}	4.14503×10^{-10}	1410.523265	16.98205462
	2	2.03472489627913	5.35822×10^{-149}	1.28956×10^{-149}	16.98205478	
	3	2.03472489627913	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.87	0.623915	0.164725		
Y5	1	2.03472489279545	1.44750×10^{-8}	3.48367×10^{-9}	11854.67132	130.407955
	2	2.03472489627913	2.54966×10^{-133}	6.13624×10^{-134}	130.4079656	
	3	2.03472489627913	$0. \times 10^{-999}$	$0. \times 10^{-999}$		

1050 significant digits, whenever its exact value is not known; in addition, the error bound $\epsilon = \frac{1}{2} \times 10^{-250}$ was used. The values of initial guess x_0 were selected close to α to guarantee convergence of iterative methods. The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is actually rounded to be accurate up to 250 significant digits, although being displayed only up to 15 significant digits. Experimental results are summarized in Tables 2–7, where bold-face numbers refer to the least error or CPU time. All experiments have been carried out on a personal computer equipped with an AMD 3.1 Ghz dual-core processor and Windows 32-bit XP operating system.

Iterative methods identified in Table 1 have been successfully applied to some test functions. Especially four test functions $f(x) = \cos\frac{\pi x}{2} + x^2 - \pi$ with $\alpha \approx 2.034724896279126$, $f(x) = x^4\cos x^2 - x^5\log(1 + x^2 - \pi) + \pi^2$ with $\alpha = \sqrt{\pi}$, $f(x) = \cos(x^2 - 2x + \frac{16}{9}) - \log(x^2 - 2x + \frac{25}{9}) - 1$ with $\alpha = 1 + \mathrm{i}\,\frac{\sqrt{7}}{3}$ and $f(x) = x^3 + \log(1 + x)$ with $\alpha = 0$ clearly demonstrated sixteenth-order convergence in Tables 2–5, respectively. Tables 2–5 list iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $|\frac{e_n}{e_{n-1}{}^{16}}|$ as well as the theoretical asymptotic error constant η. Notice that, for the test function $f(x) = x^3 + \log(1+x)$, Neta [11] did not provide the asymptotic error constants due to the precision limitation, at most with double precision capable of doing approximately 14-digit computation, on an IBM 370/148 computer system in 1981.

Convergence behavior was verified for additional test functions that are listed below:

$$f_1(x) = (2 + x^3)\cos\left(\frac{\pi x}{2}\right) + \log(x^2 + 2x + 2), \qquad \alpha = -1, \qquad x_0 = -0.93$$
$$f_2(x) = e^{-x}\cos 3x + x - 2, \qquad \alpha \approx 1.878179124117988, \qquad x_0 = 1.60$$
$$f_3(x) = x^2 e^x + x\cos\frac{1}{x^3} + 1, \qquad \alpha \approx -1.565060286750835, \qquad x_0 = -1.25$$
$$f_4(x) = xe^x + \log(1 + x + x^4), \qquad \alpha = 0, \qquad x_0 = 0.25$$
$$f_5(x) = \sin\frac{\pi}{x} - \sqrt{x^2 + x + 3} - 4x - 5, \qquad \alpha \approx -2.070657938557092, \qquad x_0 = -1.75$$
$$f_6(x) = 1 - \sin\frac{\pi(x^2 - 2x + 2)}{(x-1)^2} - \cos\frac{\pi(x^2 - 2x + 6)}{(x-1)^2}, \qquad x_0 = 0.98 - 1.38\mathrm{i}, \quad \mathrm{i} = \sqrt{-1}$$
$$f_7(x) = \cos\frac{\pi}{1 + 2x^2} \cdot \log(x^2 - 2) + x^3 - 3\sqrt{3}, \qquad \alpha = \sqrt{3}, \qquad x_0 = 1.55,$$

with $\log z$ $(z \in \mathbb{C})$ representing a principal analytic branch such that $-\pi \le \mathrm{Im}(\log z) < \pi$.

Table 6 lists the values of $|x_n - \alpha|$ within the prescribed error bound for sixteenth-order methods **N1**, **T1**, **T2** and **Y1, Y2, Y3, Y4, Y5**, where **N1** is specified with $A = 2$ in (1.5), **T1** associated with (1.3) and **T2** associated with (1.4). As Table 5 suggests, during the computational experiment for the particular choice of test functions, proposed multipoint methods show favorable performance as compared with existing methods **N1**, **T1** and **T2**. Under the same order of convergence, one should note that the speed of local convergence of $|x_n - \alpha|$ is dependent on c_j, namely $f(x)$ and α. Due to the high-order

Table 3
Convergence for $f(x) = x^4 \cos x^2 - x^5 \log(1 + x^2 - \pi) + \pi^2$ with $\alpha = \sqrt{\pi}$.

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^{16}} \rvert$	η
	0	1.65	9.74184	0.122454		
Y1	1	1.77245385090552	2.88830×10^{-13}	3.42679×10^{-15}	1.340688834	0.5498591735
	2	1.77245385090552	1.67572×10^{-230}	1.98813×10^{-232}	0.5498591735	
	3	1.77245385090552	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.65	9.74184	0.122454		
Y2	1	1.77245385090552	1.74191×10^{-13}	2.06667×10^{-15}	0.8085590752	2.711733710
	2	1.77245385090552	2.53126×10^{-233}	$\mathbf{3.00318 \times 10^{-235}}$	2.711733710	
	3	1.77245385090552	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.65	9.74184	0.122454		
Y3	1	1.77245385090552	3.39997×10^{-13}	4.03386×10^{-15}	1.578198345	1.891584067
	2	1.77245385090552	7.83632×10^{-229}	9.29731×10^{-231}	1.891584067	
	3	1.77245385090552	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.65	9.74184	0.122454		
Y4	1	1.77245385090552	2.15676×10^{-13}	2.55886×10^{-15}	1.001124104	0.8582637528
	2	1.77245385090552	2.44415×10^{-232}	2.89984×10^{-234}	0.8582637528	
	3	1.77245385090552	$0. \times 10^{-999}$	$0. \times 10^{-999}$		
	0	1.65	9.74184	0.122454		
Y5	1	1.77245385090552	2.25952×10^{-13}	2.68078×10^{-15}	1.048823548	24.35961119
	2	1.77245385090552	1.46087×10^{-230}	1.73323×10^{-232}	24.35961119	
	3	1.77245385090552	$0. \times 10^{-999}$	$0. \times 10^{-999}$		

Table 4
Convergence for $f(x) = \cos(x^2 - 2x + \frac{16}{9}) - \log(x^2 - 2x + \frac{25}{9}) - 1$ with $\alpha = 1 + \mathrm{i}\,\frac{\sqrt{7}}{3}$.

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^{16}} \rvert$	η
	0	0.9 + 0.8i	0.217980	0.129269		
Y1	1	$\begin{pmatrix}0.999999999999999\\0.881917103688197\end{pmatrix}$[a]	1.01147×10^{-15}	5.73450×10^{-16}	0.09431882768	0.2594339569
	2	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	6.25770×10^{-245}	$\mathbf{3.54778 \times 10^{-245}}$	0.2594339569	
	3	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	$0. \times 10^{-1000}$	$0. \times 10^{-999}$		
	0	0.9 + 0.8i	0.217980	0.129269		
Y2	1	$\begin{pmatrix}1.00000000000000\\0.881917103688198\end{pmatrix}$	1.99806×10^{-15}	1.13279×10^{-15}	0.1863173124	0.1302809088
	2	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	1.68945×10^{-240}	9.57829×10^{-241}	0.1302809088	
	3	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	$0. \times 10^{-1000}$	$0. \times 10^{-999}$		
	0	0.9 + 0.8i	0.217980	0.129269		
Y3	1	$\begin{pmatrix}0.999999999999993\\0.881917103688199\end{pmatrix}$	1.29683×10^{-14}	7.35231×10^{-15}	1.209278875	8.424827943
	2	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	1.08342×10^{-225}	6.14243×10^{-226}	8.424827943	
	3	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	$0. \times 10^{-1000}$	$0. \times 10^{-999}$		
	0	0.9 + 0.8i	0.217980	0.129269		
Y4	1	$\begin{pmatrix}0.999999999999997\\0.881917103688201\end{pmatrix}$	8.92948×10^{-15}	5.06254×10^{-15}	0.8326670360	2.552550763
	2	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	8.38166×10^{-229}	4.75195×10^{-229}	2.552550763	
	3	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	$0. \times 10^{-1000}$	$0. \times 10^{-999}$		
	0	0.9 + 0.8i	0.217980	0.129269		
Y5	1	$\begin{pmatrix}0.999999999999981\\0.881917103688218\end{pmatrix}$	5.05171×10^{-14}	2.86405×10^{-14}	4.710678748	24.84788094
	2	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	8.98336×10^{-216}	5.09309×10^{-216}	24.84788094	
	3	$\begin{pmatrix}1.00000000000000\\0.881917103688197\end{pmatrix}$	$0. \times 10^{-1000}$	$0. \times 10^{-999}$		

[a] $\begin{pmatrix}0.999999999999999\\0.881917103688197\end{pmatrix} = 0.999999999999999 + 0.881917103688197\,\mathrm{i},\ \mathrm{i} = \sqrt{-1}.$

convergence, it is possible to observe that all listed methods indicate $|x_3 - \alpha| < 1000$ less than the prescribed precision for the function $f_4(x)$, although it is accurate with at least 1000 digits. In Table 7, CPU times are displayed for the listed high-order iterative methods. Indeed, the CPU time of (1.1) is increased approximately by a factor of between 2 and 14, as compared with listed existing methods **N1**, **T1** and **T2**.

Table 5
Convergence for $f(x) = x^3 + \log(1+x)$ with $\alpha = 0$.

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^{16}} \rvert$	η
	0	0.25	0.238769	0.250000		
Y1	1	-2.78335×10^{-14}	2.78335×10^{-14}	2.78335×10^{-14}	0.0001195440049	0.002013507105
	2	$-2.61240 \times 10^{-220}$	2.61240×10^{-220}	2.61240×10^{-220}	0.002013507105	
	3	$-1.20896 \times 10^{-1224}$	$1.20896 \times 10^{-1224}$	$1.20896 \times 10^{-1224}$		
	0	0.25	0.238769	0.250000		
Y2	1	-3.16178×10^{-15}	3.16178×10^{-15}	3.16178×10^{-15}	0.00001357976069	0.03640386526
	2	3.63131×10^{-234}	3.63131×10^{-234}	3.63131×10^{-234}	0.03640386526	
	3	$1.78944 \times 10^{-1243}$	$1.78945 \times 10^{-1243}$	$1.78945 \times 10^{-1243}$		
	0	0.25	0.238769	0.250000		
Y3	1	2.95577×10^{-16}	2.95578×10^{-16}	2.95578×10^{-16}	0.00000126949675	0.01604134516
	2	$-5.44488 \times 10^{-251}$	5.44488×10^{-251}	$\mathbf{5.44488 \times 10^{-251}}$	0.01604134516	
	3	$-3.78875 \times 10^{-1253}$	$3.78875 \times 10^{-1253}$	$3.78875 \times 10^{-1253}$		
	0	0.25	0.238769	0.250000		
Y4	1	-1.09681×10^{-14}	1.09681×10^{-14}	1.09681×10^{-14}	0.00004710770408	0.003440574363
	2	1.50919×10^{-226}	1.50919×10^{-226}	1.50919×10^{-226}	0.003440574363	
	3	$3.51392 \times 10^{-1234}$	$3.51392 \times 10^{-1234}$	$3.51392 \times 10^{-1234}$		
	0	0.25	0.238769	0.250000		
Y5	1	-1.44466×10^{-13}	1.44466×10^{-13}	1.44466×10^{-13}	0.0006204778908	0.007892071759
	2	$-2.84088 \times 10^{-208}$	2.84089×10^{-208}	2.84089×10^{-208}	0.007892071759	
	3	$-1.10294 \times 10^{-1214}$	$1.10295 \times 10^{-1214}$	$1.10295 \times 10^{-1214}$		

Table 6
Comparison of $\lvert x_n - \alpha\rvert$ for high-order iterative methods.

f	x_0	$\lvert x_n - \alpha\rvert$	**N1**	**T1**	**T2**	**Y1**	**Y2**	**Y3**	**Y4**	**Y5**
f_1	−0.93	$\lvert x_1 - \alpha\rvert$	4.82e−10[a]	2.24e−8	1.83e−10	1.44e−10	1.66e−9	2.68e−9	3.05e−9	1.43e−8
		$\lvert x_2 - \alpha\rvert$	7.56e−138	1.31−108	2.77e−145	**8.08e−147**	5.07e−128	3.74e−124	3.74e−123	1.46e−111
		$\lvert x_3 - \alpha\rvert$	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000
f_2	1.60	$\lvert x_1 - \alpha\rvert$	4.07e−12	3.26e−10	1.33e−12	1.16e−12	3.20e−10	8.99e−10	9.52e−10	6.99e−9
		$\lvert x_2 - \alpha\rvert$	7.95e−184	1.34e−152	1.59e−192	**1.67e−193**	1.50e−151	3.80e−143	8.75e−143	6.21e−128
		$\lvert x_3 - \alpha\rvert$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_3	−1.25	$\lvert x_1 - \alpha\rvert$	1.86e−12	4.61e−9	2.49e−11	5.84e−13	2.25e−11	1.17e−10	2.83e−12	1.49e−10
		$\lvert x_2 - \alpha\rvert$	4.19e−194	2.97e−137	5.79e−176	**2.58e−201**	1.01e−176	1.92e−164	5.11e−191	1.24e−163
		$\lvert x_3 - \alpha\rvert$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_4	0.25	$\lvert x_1 - \alpha\rvert$	9.91e−14	3.01e−11	4.54e−14	4.29e−14	5.89e−13	2.74e−12	1.38e−14	6.37e−12
		$\lvert x_2 - \alpha\rvert$	6.28e−215	7.12e−171	1.85e−219	8.70e−221	1.34e−202	6.73e−192	**9.17e−229**	1.43e−186
		$\lvert x_3 - \alpha\rvert$	2.60e−1225	1.26e−1177	5.57e−1225	3.64e−1225	3.39e−1205	8.05e−1196	2.75e−1234	4.71e−1196
f_5	−1.75	$\lvert x_1 - \alpha\rvert$	6.06e−16	9.68e−17	3.66e−16	4.33e−17	2.04e−15	4.78e−16	2.68e−15	7.75e−15
		$\lvert x_2 - \alpha\rvert$	1.63e−254	1.42e−270	4.50e−258	**6.41e−273**	9.07e−246	3.03e−256	7.21e−244	2.68e−236
		$\lvert x_3 - \alpha\rvert$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_6	0.98	$\lvert x_1 - \alpha\rvert$	5.67e−15	9.22e−7	6.37e−15	4.84e−15	1.74e−14	5.30e−15	1.45e−14	3.15e−14
	−1.38i	$\lvert x_2 - \alpha\rvert$	8.09e−222	1.91e−184	4.37e−221	**2.15e−223**	2.48e−213	1. 88e−222	1.63e−214	2.10e−208
		$\lvert x_3 - \alpha\rvert$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_7	1.55	$\lvert x_1 - \alpha\rvert$	2.47e−12	6.16e−1	5.42e−12	7.12e−12	1.11e−12	9.11e−12	8.49e−13	8.94e−12
		$\lvert x_2 - \alpha\rvert$	8.97e−190	1.77e−5	2.84e−184	6.70e−182	1.16e−194	3.38e−180	**1.49e−196**	4.13e−180
		$\lvert x_3 - \alpha\rvert$	0.e−999	4.41e−71	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999

[a] 4.82e−10 denotes 4.82×10^{-10}.

Table 7
Comparison of CPU times for high-order methods.

f	x_0	CPU time (s)							
		N1	**T1**	**T2**	**Y1**	**Y2**	**Y3**	**Y4**	**Y5**
f_1	−0.93	141.344	295.781	28.875	**10.406**	13.375	11.156	13.516	11.328
f_2	1.60	184.109	176.594	35.688	**13.547**	16.61	13.984	16.594	14.000
f_3	−1.25	375.813	570.625	76.609	**27.063**	33.687	27.797	33.703	27.844
f_4	0.25	129.094	431.235	40.25	**14.359**	18.437	15.500	17.735	15.656
f_5	−1.75	91.296	184.032	18.671	**6.766**	12.625	6.890	12.594	10.391
f_6	0.98 − 1.38i	423.922	936.766	86.719	**30.612**	38.953	31.875	38.859	32.406
f_7	1.55	134.687	537.141	27.219	**9.968**	12.141	10.266	12.078	10.234

Although limited to the particular test functions chosen in these numerical experiments, **Y1** has shown best accuracy for f_1, f_2, f_3, f_5, f_6, while **Y4** for f_4, f_7. In general, one should note that the computational accuracy strongly depends on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. The corresponding efficiency index for the proposed family of methods (1.7) is $16^{1/5} \approx 1.741101$ better than $8^{1/4} \approx 1.68179$ of optimal eighth-order methods. Iterative scheme (1.7) is believed to be more favorable than other listed methods due to fast computational time and acceptable accuracy. An iterative method with optimal convergence order of 32 satisfying the conjecture of Kung–Traub [10] can be developed by considering an additional fifth-step weighting function in (1.7) and extending the current analysis presented here.

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